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Universal Properties of Infinite Matrices

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1. INTRODUCTION

It is well known that the rings of $n \times n$ matrices have the "universal" properties that every associative algebra of dimension n over a field F embeds in $M_n(F)$ and every semigroup (with unit) of order n is isomorphic to a semigroup of $n \times n$ $(0,1)$-matrices. In each case, the result follows easily by considering the regular representation. Similarly, every associative F-algebra of countable dimension has a faithful representation in the algebra of row-finite countably infinite matrices over F, and every countable semigroup can be represented by row-finite countable $(0,1)$-matrices. (In fact, "row-finite" can be replaced by "row- and column-finite" [2, p. 413, Proposition 2.1].)

In this paper we establish corresponding results for non-associative algebras and groupoids, using infinite matrices without row or column finiteness. Here the situation is different, in that it is not possible to use the regular representation, for which associativity is critical; also, these matrices form only a partial ring.

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More precisely, for any ring R, let $M_\infty(R)$ denote the set of countable square matrices over R indexed by $\mathbb{N} = \{1, 2, 3, \dots\}$, i.e.,

$$M_\infty(R) = \left\{A = \left(A_{ij}\right) : A_{ij} \in R \text{ and } i, j \in \mathbb{N}\right\}.$$

Clearly $M_\infty(R)$ is an additive group in the natural way, and for some pairs of matrices $A, B \in M_\infty(R)$ the product AB can be defined (as usual) by

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}.$$

For our purposes the product AB is defined only if all such sums are finite, i.e., only if for all i, j the set $\{k\colon A_{ik} B_{kj} \neq 0\}$ is finite. Note that even when R is associative and all products concerned are defined, $(AB)C$ and $A(BC)$ need not be equal; a simple example is given by

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{pmatrix}.$$

Thus $M_\infty(R)$ forms a partial non-associative ring. If R has a unit then the unit matrix in $M_\infty(R)$ will be written as I.

In Section 2 we establish the technical result from which most of our other results follow rather straightforwardly. This essentially states that in $M_\infty(R)$ certain sets of quadratic expressions can be made equal to whatever values are required.

Section 3 contains the results that every non-associative F-algebra of finite or countable dimension and every finite or countable groupoid has a faithful representation in $M_\infty(F)$. There are also corresponding results (see Section 5) for arbitrary F-algebras and groupoids (i.e., without any cardinality restrictions).

Section 4 explores the solvability of systems of equations of the form $f_j = P_j$ $(j \in J)$, where the f_j are non-associative polynomials over F and the $P_j \in M_\infty(F)$. The case where all $P_j = 0$ has a straightforward answer; we also consider when such systems can be solved for arbitrary P_j. Some further related problems are discussed in Section 5.

Fields are always assumed to be associative and commutative, whereas rings and algebras (partial or otherwise) are allowed to be non-commutative and non-associative, and need not, except where specified, have a 1.

Apart from (a) the well-known facts noted above about embeddings of associative systems in finite or row-finite matrices, (b) sporadic remarks in the literature (see, e.g., [1, pp. 8–9], [5, p. 5], [6, pp. 122–123]) about the non-associativity of $M_\infty(R)$, and (c) results about embedding special classes of non-associative systems into better-behaved subclasses, we have not been able to trace any previous work having a substantial connection with our results here.

2. SOLVABILITY OF QUADRATIC SYSTEMS

THEOREM 1. *Let R be a ring with 1, and let $A_{pq} \in M_\infty(R)$, $\alpha_{pqr} \in R$ $(p, q, r \in \mathbb{N})$ be such that the set $\{r\colon \alpha_{pqr} \neq 0\}$ is finite for all p, q. Then there exist linearly independent matrices $X_r \in M_\infty(R)$ $(r \in \mathbb{N})$ such that for all $p, q \in \mathbb{N}$ the product $X_p X_q$ is defined and*

$$X_p X_q = A_{pq} + \sum_r \alpha_{pqr} X_r. \tag{1}$$

Proof. Since $\{(i, j, p, q) : i, j, p, q \in \mathbb{N}\} = \mathbb{N}^4$ is countable, there is a bijection $\mathbb{N} \to \mathbb{N}^4$, say $m \mapsto (i_m, j_m, p_m, q_m)$. Let $s_1, s_2, \ldots$ be a strictly increasing sequence of integers such that $s_m > \max(i_m, j_m)$, and define

$$X_r = \sum_u \left[\delta(r, p_u) E(i_u, s_u) + \delta(r, q_u) y_u E(s_u, j_u) \right],$$

where $E(i, j) \in M_\infty(R)$ denotes the matrix with 1 in the (i, j) place and zeros elsewhere, δ is the Kronecker symbol, and $y_1, y_2, \ldots$ are elements of R to be specified later. Clearly each X_r is a well-defined element of $M_\infty(R)$. If $\sum \beta_r X_r = 0$ then $0 = (\sum \beta_r X_r)_{i_n, s_n} = \sum \beta_r \delta(r, p_n) = \beta_{p_n}$; since the p_n take on every value, it follows that the X_r are linearly independent.

We shall show that $X_p X_q$ is well defined (whatever the values of y_u), and that we can choose these y_u so that (1) holds. Now

$$\begin{aligned}(X_{p_n} X_{q_n})_{i_n, j_n} &= \textstyle\sum_k (X_{p_n})_{i_n, k} (X_{q_n})_{k, j_n} \\ &= \sum_k \Big\{ \sum_u \left[\delta(p_n, p_u) \delta(i_u, i_n) \delta(s_u, k) \right. \\ &\qquad \left. + \delta(p_n, q_u) \delta(s_u, i_n) \delta(j_u, k) y_u \right] \\ &\qquad \times \sum_v \left[\delta(q_n, p_v) \delta(i_v, k) \delta(s_v, j_n) \right. \\ &\qquad \left. + \delta(q_n, q_v) \delta(s_v, k) \delta(j_v, j_n) y_v \right] \Big\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{u,v} \delta(p_n, p_u)\,\delta(i_u, i_n)\,\delta(q_n, p_v)\,\delta(i_v, s_u)\,\delta(s_v, j_n) \\
&\quad + \sum_{u,v} \delta(p_n, p_u)\,\delta(i_u, i_n)\,\delta(q_n, q_v)\,\delta(s_v, s_u)\,\delta(j_v, j_n)\,y_v \\
&\quad + \sum_{u,v} \delta(p_n, q_u)\,\delta(s_u, i_n)\,\delta(q_n, p_v)\,\delta(i_v, j_u)\,\delta(s_v, j_n)\,y_u \\
&\quad + \sum_{u,v} \delta(p_n, q_u)\,\delta(s_u, i_n)\,\delta(q_n, q_v)\,\delta(s_v, j_u)\,\delta(j_v, j_n)\,y_u y_v \\
&= A + B + C + D \qquad \text{(say)}.
\end{aligned}
$$

Nonzero terms can occur in A only when $s_v = j_n$, which can hold for at most one v, after which $s_u = i_v$ can hold for at most one u. Hence $A = 0$ or 1.

Nonzero terms can occur in B only when $s_u = s_v$ (so $u = v$) and $p_n = p_u$, $i_n = i_u$, $q_n = q_v = q_u$, $j_n = j_v = j_u$ (so $n = u$). Hence there is exactly one nonzero term in the sum, when $u = v = n$, and so $B = y_n$.

Nonzero terms can occur in C or D only when $s_u = i_n$, which can hold for at most one u (also $s_u = i_n < s_n$, so $u < n$). For C, nonzero terms also require $s_v = j_n$, which can hold for at most one v; hence $C = 0$ or y_u (for some $u < n$). Similarly for D we need $s_v = j_u$, which can hold for at most one v (also $s_v = j_u < s_u$, so $v < u$); hence $D = 0$ or $y_u y_v$ (for some $v < u < n$).

Thus $(X_{p_n} X_{q_n})_{i_n, j_n}$ is well defined (it is a finite sum) and

$$
(X_{p_n} X_{q_n})_{i_n, j_n} = A + y_n + C + D = y_n + Q(y_1, \ldots, y_{n-1}), \tag{2}
$$

where Q is either zero or a sum of one or more of the three terms 1, y_u, $y_u y_v$ (for some $v < u < n$).

It is easy to see that the (i_n, j_n) entry of X_r is either 0, 1, or y_u for some $u < n$. Since only finitely many α_{pqr} are nonzero for each p, q, it follows that

$$
\left[A_{p_n q_n} + \sum_r \alpha_{p_n q_n r} X_r \right]_{i_n, j_n} = L(y_1, \ldots, y_{n-1}), \tag{3}
$$

i.e., some linear expression in $y_1, \ldots, y_{n-1}$.

Comparing (2) and (3) we see that, to ensure that

$$
(X_{p_n} X_{q_n})_{i_n, j_n} = \left[A_{p_n q_n} + \sum_r \alpha_{p_n q_n r} X_r \right]_{i_n, j_n}
$$

for all n, it suffices to define y_n inductively as

$$y_n = L(y_1, \ldots, y_{n-1}) - Q(y_1, \ldots, y_{n-1}) \qquad (n \in \mathbb{N}).$$

Since $\{(i_n, j_n, p_n, q_n) : n \in \mathbb{N}\} = \mathbb{N}^4$, it then follows that

$$X_p X_q = A_{pq} + \sum_r \alpha_{pqr} X_r$$

for all p, q. ∎

REMARKS. (i) In particular, any (e.g., finite) subcollection of conditions of type (1) can be satisfied.

(ii) We note some special cases, none of which seems to have been previously recorded.

(a) Every matrix $A \in M_\infty(R)$ has a square root in $M_\infty(R)$. This contrasts with the situation for finite matrices, where, for example, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ does not have a square root in $M_2(R)$ for any commutative associative ring R.

(b) There exist X_1, X_2 in $M_\infty(R)$ such that $X_1 X_2 = 0$, $X_2 X_1 = I$. By contrast, clearly it is impossible to have $a_1 a_2 = 0$, $a_2 a_1 = 1$ for a_1, a_2 in any (nonzero) associative system A (in particular, in the ring of row-finite matrices over any associative ring).

(c) There exist X_1, X_2 in $M_\infty(R)$ such that $X_1 X_2 = X_2 X_1 = I$, $X_1^2 = X_2^2 = 0$; thus $M_\infty(R)$ contains invertible nilpotent matrices.

3. REPRESENTATIONS IN M_∞

THEOREM 2. *Let F be a field and A any (non-associative) F-algebra of finite or countable dimension over F. Then A embeds in $M_\infty(F)$.*

Moreover, if A has a 1, *this embedding can be chosen so that* $1 \mapsto I$.

Proof. Let A have F-basis $\{a_1, a_2, \ldots\}$ with $a_p a_q = \sum \alpha_{pqr} a_r$. By Theorem 1 (with all $A_{pq} = 0$), there exist linearly independent $X_r \in M_\infty(F)$ such that $X_p X_q = \sum \alpha_{pqr} X_r$. Now the map ϕ: $A \to M_\infty(F)$ given by

$$\phi(\Sigma \beta_r a_r) = \Sigma \beta_r X_r$$

clearly defines an injective F-algebra homomorphism from A to $M_\infty(F)$.

For the case where A has a 1, take a basis of the form $\{1, a_1, a_2, \ldots\}$, say with

$$a_p a_q = \gamma_{pq} 1 + \sum \alpha_{pqr} a_r \qquad (p, q \in \mathbb{N}).$$

Theorem 1 gives us linearly independent $X_r \in M_\infty(F)$ $(r \in \mathbb{N})$ such that

$$X_p X_q = \gamma_{pq} I + \sum \alpha_{pqr} X_r \qquad (p, q \in \mathbb{N});$$

the form of the X_r in the proof ensures that $\{I, X_1, X_2, \dots\}$ remains linearly independent. Now $\beta 1 + \sum \beta_r a_r \mapsto \beta I + \sum \beta_r X_r$ gives the required embedding. ∎

With appropriate natural definitions, there is a corresponding result for any ring A which is a finitely or countably generated free left R-module, where R is any commutative and associative ring with 1.

THEOREM 3. *Let G be any finite or countable groupoid. Then G embeds in $M_\infty(R)$ for any ring R with* 1.

Proof. Again this follows immediately from Theorem 1. ∎

4. POLYNOMIALS AND MONOMIALS

Let $V = \{v_1, v_2, \dots\}$ be a non-empty finite or countable set of variables and let $F\{V\}$ denote the free non-associative algebra on V over a field F. (Thus $F\{V\}$ has an F-basis consisting of all the non-associative monomials in V, with the multiplication defined by juxtaposition and F-linearity, and with the empty monomial as unit element; for a general treatment, see, for example, [3, pp. 78–83].)

Throughout this section, all F-algebras, partial or otherwise, are nonzero with 1 (and subalgebras have the same 1 as the containing algebra and hence are nonzero).

Let f_j $(j \in J)$ be elements of $F\{V\}$ and consider the system

$$f_j = 0 \qquad (j \in J). \tag{4}$$

Let M be any partial F-algebra. We say that the system (4) is *solvable* in M if there is a subalgebra A of M and elements $a_i \in A$ such that $f_j(a_1, a_2, \dots) = 0$ $(j \in J)$. The system (4) is called *consistent* if it is solvable in some partial F-algebra; this of course is equivalent to saying it is solvable in some full (as opposed to partial) F-algebra.

PROPOSITION 4. *Let U be any partial F-algebra which contains an isomorphic copy of every countably generated F-algebra* (*equivalently, of every F-algebra of countable dimension*). *Note that, by Theorem* 2, *$M_\infty(F)$ is one such U. Then the following are equivalent*:

(i) *the system* (4) *is consistent*;

(ii) *the system* (4) *is solvable in U*;

(iii) *the ideal (f_j) generated by the f_j is proper, i.e., $(f_j) \neq F\{V\}$.*

Proof. (i) ⇒ (ii). Since (4) is consistent, there exists an F-algebra A in which there is a solution $v_i = a_i$. Taking the subring of A generated over F by the a_i, we obtain a countably generated F-algebra B in which the system is solvable. Then, by hypothesis, B embeds in U, so (4) is solvable in U.

(ii) ⇒ (iii). Let $v_i = a_i$ be a solution of (4), where $a_i \in A$ (some subalgebra of U). Suppose (f_j) is not proper, so $1 \in (f_j)$. Then for suitable terms $t_k \in F\{V\}$, each having some f_j as a factor, and suitable $\alpha_k \in F$, we have $\sum \alpha_k t_k = 1$. But then, in A, $0 = \sum \alpha_k t_k(a_1, a_2, \dots) = 1$, a contradiction.

(iii) ⇒ (i). Suppose (f_j) is proper. Then $A = F\{V\}/(f_j)$ is an F-algebra in which (4) has a solution, namely $\bar{v}_i$. ∎

In our context, the main significance of Proposition 4 is that the system (4) is solvable in $M_\infty(F)$ if and only if it is consistent.

Note that, with suitable modifications, Proposition 4 holds for systems of equations $f_j = g_j$ in the free groupoid; in particular (iii) must be replaced by the condition that the congruence generated by the (f_j, g_j) $(j \in J)$ is proper.

Proposition 4 deals with solving systems of the form $f_j = 0$. More generally, consider next the system

$$f_j = p_j \qquad (j \in J), \tag{5}$$

where f_j $(j \in J)$ are elements of $F\{V\}$, M is a partial F-algebra, and $p_j \in M$ $(j \in J)$. We say that system (5) is *quasi-solvable* in M if there exist $a_i \in M$ such that $f_j(a_1, a_2, \dots)$ is well defined and equal to p_j $(j \in J)$; in this case $(a_1, a_2, \dots)$ is called a *quasi-solution*. (Whereas for solvability of (4), the a_i were required to lie in some subalgebra of M, for quasi-solvability this condition is dropped. As a simple example of the difference this makes, note that from Remark (ii)(a) the equation $X^2 = P$ is always quasi-solvable in $M_\infty(R)$, but it is not solvable if P^2 is undefined.)

We call an element $f \in F\{V\}$ *assignable* in M if, for all $p \in M$, the equation $f = p$ is quasi-solvable in M. A set of elements f_j $(j \in J)$ in $F\{V\}$ is called *independently assignable* in M if for all $p_j \in M$ the system (5) is quasi-solvable in M.

We consider which sets are independently assignable in $M_\infty(F)$. The simplest case is that of monomials. Clearly not every set of monomials is independently assignable: e.g., if $x^2 = 0$ then $x^2 x$ cannot be nonzero. (This also shows that the system $f_j = 0$ $(j \in J)$ can be consistent without the f_j being independently assignable.) We define a set S of monomials to be *unembedded* if $1 \notin S$ and there are no $\sigma, \tau \in S$ such that σ is a proper submonomial of τ.

THEOREM 5. *Let V be a finite or countable set, and S a non-empty set of monomials in V. Then S is independently assignable in $M_\infty(F)$ (or indeed in $M_\infty(R)$ for any ring R with 1) if and only if S is unembedded.*

Proof. If $1 \in S$ the theorem becomes obvious, so assume $1 \notin S$.

($\Rightarrow$) If S is not unembedded, i.e., if σ is a proper submonomial of τ for some $\sigma, \tau \in S$, then clearly even the subsystem $\sigma = 0$, $\tau = I$ has no quasi-solution.

($\Leftarrow$) If S is unembedded, then without loss of generality all $\sigma \in S$ have degree 2 or more, and every $v \in V$ occurs in some monomial in S. Let $S = \{\sigma_j : j \in J\}$, and let $P_j \in M_\infty(F)$ $(j \in J)$. We must find X_r such that $\sigma_j(X_1, X_2, \dots) = P_j$ $(j \in J)$.

The set W of all proper submonomials, other than 1, of elements of S contains V and can be enumerated as $\tau_1, \tau_2, \dots$ in such a way that if τ_i is a submonomial of τ_j then $i < j$. (Explicitly, let W_d be the set of those elements of W which are monomials in $v_1, \dots, v_d$ of total degree at most d; now order $W = \bigcup(W_{d+1} \setminus W_d)$ by increasing d and then by taking each $W_{d+1} \setminus W_d$ in some order of non-decreasing degree.) Let $v = \tau_{n_v}$ $(v \in V)$ and let $T = \{n_v \colon v \in V\}$. Then $T \subseteq \mathbb{N}$, and, for every $k \in \mathbb{N} \setminus T$, we have $\tau_k = \tau_{p_k}\tau_{q_k}$ for some $p_k, q_k < k$. Moreover, each $\sigma_j \in S$ is of the form $\sigma_j = \tau_{r_j}\tau_{s_j}$ for suitable (unique) r_j, s_j. By Theorem 1, there exist $X_r \in M_\infty(F)$ such that

$$X_{p_k}X_{q_k} = X_k \qquad (k \in \mathbb{N} \setminus T)$$

and

$$X_{r_j}X_{s_j} = P_j \qquad (j \in J).$$

(Note that the hypothesis that S is unembedded ensures that W and S are disjoint, so all the left-hand sides of the above system are distinct.) Now $v = X_{n_v}$ $(v \in V)$ is the required quasi-solution. ∎

As a very special case of Theorem 5, given arbitrary B and C in $M_\infty(R)$ one can find an X such that $X^2X = B$, $XX^2 = C$.

THEOREM 6. *Every non-constant $f \in F\{V\}$ is assignable in $M_\infty(F)$.*

Proof. Let f have (positive) degree n, and let σ be a monomial of degree n appearing in f with nonzero coefficient; without loss of generality we may take this coefficient to be 1. Only a finite number of variables can occur in f, say $v_1, \dots, v_t$. For any given $P \in M_\infty(F)$ we must find $X_1, \dots, X_t \in M_\infty(F)$ such that $f(X_1, \dots, X_t) = P$. Let S be the set of all monomials in $v_1, \dots, v_t$ of total degree at most n. Enumerate S as $\{\tau_0, \tau_1, \tau_2, \dots, \tau_u\}$ in order of non-decreasing degree, with $\tau_0 = 1$, $\tau_j = v_j$ $(j = 1, \dots, t)$ and $\tau_u = \sigma$ (since we may suppose $n \geq 2$).

Then for $t < k \le u$ we have $\tau_k = \tau_{p_k}\tau_{q_k}$ for suitable (unique) positive $p_k, q_k < k$, and we may write $f(x) = (\sum_{k=0}^{u-1} \beta_k \tau_k) + \sigma$, where each $\beta_k \in F$. By Theorem 1, there exist matrices $X_1, \ldots, X_{u-1} \in M_\infty(F)$ such that

$$X_{p_k} X_{q_k} = X_k \qquad (t < k < u)$$

and

$$X_{p_u} X_{q_u} = (P - \beta_0 I) - \sum_{k=1}^{u-1} \beta_k X_k.$$

Thus $f(X_1, \ldots, X_t) = \beta_0 I + \sum_{k=1}^{u-1} \beta_k X_k + X_{p_u} X_{q_u} = P$. ∎

COROLLARY 7. *$M_\infty(F)$ satisfies no non-trivial polynomial identity. More precisely, there is no non-trivial non-associative polynomial $f(v_1, \ldots, v_t)$ such that $f(X_1, \ldots, X_t) = 0$ in $M_\infty(F)$ whenever it is defined.*

5. COMMENTS

Theorem 2 can be generalized to algebras of uncountable dimension. For any set Λ we can define $M_\Lambda(F)$ to be the partial F-algebra consisting of all square matrices over F whose rows and columns are indexed by Λ, where (as before) products are defined only if all sums involved are finite. Up to isomorphism, $M_\Lambda(F)$ depends only on the cardinality of Λ, and this cardinality may be identified with the corresponding initial ordinal, say λ, so we may write $M_\lambda(F)$ and regard Theorem 2 as concerning the case $\lambda = \omega_0 = \aleph_0$. For arbitrary infinite cardinals γ and λ, consider the statement

$S(\gamma, \lambda)$: *every F-algebra of dimension γ can be embedded in $M_\lambda(F)$.*

By Theorem 2, $S(\aleph_0, \aleph_0)$ is true. In fact $S(\gamma, \gamma)$ holds for every infinite cardinal γ, since the method of proof of Theorem 1 extends without much difficulty. The set of all ordered quadruples of ordinals less than γ is arranged in a transfinite sequence of type γ,

$$(i_\mu, j_\mu, p_\mu, q_\mu) \qquad (0 \le \mu < \gamma), \tag{6}$$

and s_μ is defined inductively as the least ordinal greater than i_μ, j_μ, and all s_ν $(\nu < \mu)$. The argument requires that $s_\mu < \gamma$ for all μ, and this is satisfied automatically when γ is a regular cardinal (see, e.g., [4, p. 27]); it can also be guaranteed for arbitrary γ by, for example, ensuring that, in the sequence (6), quadruples (i, j, p, q) with a smaller value of $\max(i, j, p, q)$ precede those with a larger value. The rest of the proof proceeds as before, with transfinite induction replacing ordinary induction.

Thus, for the truth of $S(\gamma, \lambda)$, it is sufficient that $\lambda \geq \gamma$, while cardinality considerations show that $2^{\lambda} \geq \gamma$ is necessary; there is an obvious gap, but we do not even know whether $S(\aleph_1, \aleph_0)$ is true.

Naturally, similar considerations apply to the embedding of groupoids.

With regard to Section 4, an obvious question is what can be said about whether or not a given set of polynomials f_j is independently assignable. Since there appears to be no known algorithm even for deciding whether a given set of polynomials generates a proper ideal of $F\{V\}$ (even in the associative case), this seems to be a hard question in general. However, the following two cases can be settled.

(i) $f_1 = v_1 + v_2$, $f_2 = v_1 v_2$ are independently assignable in $M_\infty(F)$ (this can be shown using a proof rather similar to that for Theorem 1).

(ii) Let $f_1, \ldots, f_m \in F\{V\}$ all be of degree n, and write $f_j = \sum \alpha_{jk} \sigma_k +$ terms of lower degree, where σ_k are all the monomials of degree n. Then if the matrix (α_{jk}) has rank m, the set $f_1, \ldots, f_m$ is independently assignable. (The proof reduces to an argument parallel to that for Theorem 6.)

There is a more general version of independent assignability. Let M be any partial F-algebra, and consider sets of polynomials f_j in the free algebra $F\{V; W\}$ in two sets of variables. We can now ask whether there exist $a_i \in M$ such that for all $p_j \in M$, there exist $b_k \in M$ such that $f_j(a_1, a_2, \ldots; b_1, b_2, \ldots) = p_j$ for all j. In general this is harder than the question considered above. There is one case that can be proved when $M = M_\infty(F)$: the system $f_1 = vw$, $f_2 = wv$. Here we take v to be any $(0, 1)$-matrix A satisfying

(i) there are infinitely many unit entries in each row and column; and

(ii) for each $s = 1, 2, \ldots,$ in the "L-shape" of positions $(1, s), (2, s), \ldots, (s, s), (s, s-1), \ldots, (s, 1)$ there is at most one nonzero entry.

Then for any $P_1, P_2 \in M_\infty(F)$ there exists $X \in M_\infty(F)$ such that $AX = P_1$, $XA = P_2$. The proof is by methods parallel to those of Theorem 1.

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